Indag. Mathem., N.S., 5 (3), 253–257 September 26, 1994

Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts

by Annette A'Campo-Neuen

Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, Schweiz

Communicated by Prof. T.A. Springer at the meeting of June 21, 1993

ABSTRACT

Based on a counterexample to Hilbert's fourteenth problem, given by P. Roberts, we construct an example of a linear $(\mathbb{G}_a)^{12}$-action such that the algebra of invariants is not finitely generated.

INTRODUCTION

In [R] P. Roberts gives a new counterexample to Hilbert's fourteenth problem, namely the question whether for a subfield L of the function field $F(x_1, \ldots, x_n)$ in n variables over a ground field F such that $L \supset F$ the F-algebra $B := L \cap F[x_1, \ldots, x_n]$ is finitely generated. Roberts' construction is completely different from the approach of M. Nagata ([N]) whose famous counterexample has been modified and considerably simplified by R. Steinberg ([S], compare also [LT]).

However, Roberts' example does not involve a group action whereas Hilbert's motivation for the fourteenth problem was the special case where the subfield L is the quotient field $\mathrm{Quot}(F[x_1, \ldots, x_n]^G)$ of the ring of invariants of a group G acting linearly on the n-dimensional vector space over F and $B = F[x_1, \ldots, x_n]^G$.

We will first show that one can easily view the algebra B, considered by Roberts, as the ring of invariants of a non-linear action of the additive group $\mathbb{G}_a$ of the ground field F. Then we construct a linear action whose ring of invariants turns out to be a polynomial ring in one variable over B, thereby obtaining a new counterexample also for the special case of the fourteenth problem.

1. ROBERTS' COUNTEREXAMPLE AS RING OF INVARIANTS OF A NON-LINEAR $\mathbb{G}_a$-ACTION ON $\mathbb{A}^7$

For the construction of the counterexample Roberts considers the polynomial algebra $R = F[X, Y, Z, S, T, U, V]$ in 7 variables over a not necessarily algebraically closed field F of characteristic 0 with the grading determined by assigning to X, Y, Z the degree 0 and to S, T, U, V the degree 1.

In R considered as R_0-module the elements S, T, U, V generate a free R_0-submodule. Choosing a natural number $m \in \mathbb{N}$, $m \geq 2$, Roberts defines on this submodule an R_0-module homomorphism $f : R_0\, S \oplus R_0\, T \oplus R_0\, U \oplus R_0\, V \to R_0$ by $f(S) = X^{m+1}, f(T) = Y^{m+1}, f(U) = Z^{m+1}, f(V) = (XYZ)^m$.

The submodule $\operatorname{Ker} f$ generates a subalgebra of R which is denoted by A. In his article [R] Roberts shows, that the F-algebra $B := R \cap \operatorname{Quot}(A)$ is not finitely generated. In particular, B is a new counterexample to Hilbert's fourteenth problem which is even defined over $\mathbb{Q}$.

Roberts also gives the following different description of the algebra B. Namely, the homogeneous components of B with respect to the grading defined above satisfy $B_n = \operatorname{Ker} \Phi_n$, where

$$\Phi_n : R_n \to R_{n-1}, \quad m_1 \cdots m_n \mapsto \sum_{j=1}^{n} m_1 \cdots \Phi_1(m_j) \cdots m_n \qquad (m_j \in R_1)$$

is the R_0-linear map which is determined by $\Phi_0(R_0) = 0$ and by $\Phi_1(S) = X^{m+1}$, $\Phi_1(T) = Y^{m+1}, \Phi_1(U) = Z^{m+1}$ and $\Phi_1(V) = (XYZ)^m$.

This means that $B = \{f \in R \mid Df = 0\}$, where D denotes the R_0-derivation of R, defined by $D(S) = X^{m+1}$, $D(T) = Y^{m+1}$, $D(U) = Z^{m+1}$ and $D(V) = (XYZ)^m$. As one can easily verify the F-derivation D is locally nilpotent and induces therefore an algebraic action of the additive group $\mathbb{G}_a$ of F on the 7-dimensional affine space $\mathbb{A}^7$ over F, namely

$$\mathbb{G}_a \times \mathbb{A}^7 \to \mathbb{A}^7, \quad (\lambda, p) \mapsto \exp(\lambda D)(p) = p + \lambda D(p).$$

We identify $\mathbb{A}^7$ with the underlying vector space and denote the coordinates of a point p in such a way that a small letter refers to the value of the coordinate function written with the corresponding capital letter. In coordinates the action is the following:

$$(\lambda, p) = \left(\lambda, \begin{pmatrix} x \\ y \\ z \\ s \\ t \\ u \\ v \end{pmatrix} \right) \mapsto \begin{pmatrix} x \\ y \\ z \\ s + \lambda x^{m+1} \\ t + \lambda y^{m+1} \\ u + \lambda z^{m+1} \\ v + \lambda (xyz)^m \end{pmatrix}.$$

The ring of invariants of this $\mathbb{G}_a$-action is precisely B.

2. CONSTRUCTION OF A LINEAR $(\mathbb{G}_a)^{12}$-ACTION WHOSE ALGEBRA OF INVARIANTS IS NOT FINITELY GENERATED

We consider the action of the commutative group $H = (\mathbb{G}_a)^{12}$ on the 19-dimensional vectorspace V over F induced by the following group homomorphism of H into the matrix group $\mathrm{GL}_{19}(F)$, where void entries stand for zero entries:

$$
\begin{array}{ccc}
H \longrightarrow & & \mathrm{GL}_{19}(F) \\
(\lambda, \mu_1, \ldots, \mu_{11}) \longmapsto & &
\left(\begin{array}{cccc|ccc|ccc|ccc|cccccc}
1&&&&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&&&&& \\
\hline
\mu_1&\lambda&&&1&&&&&&&&&&&&&& \\
\mu_2&\mu_1&&&&1&&&&&&&&&&&&& \\
&\mu_2&&&&&1&&&&&&&&&&&& \\
\hline
\mu_3&&\lambda&&&&&1&&&&&&&&&&& \\
\mu_4&&\mu_3&&&&&&1&&&&&&&&&& \\
&&\mu_4&&&&&&&1&&&&&&&&& \\
\hline
\mu_5&&&\lambda&&&&&&&1&&&&&&&& \\
\mu_6&&&\mu_5&&&&&&&&1&&&&&&& \\
&&&\mu_6&&&&&&&&&1&&&&&& \\
\hline
\mu_7&\lambda&&&&&&&&&&&&1&&&&& \\
\mu_8&\mu_7&&&&&&&&&&&&&1&&&& \\
\mu_9&&\mu_8&&&&&&&&&&&&&1&&& \\
\mu_{10}&&\mu_9&&&&&&&&&&&&&&1&& \\
\mu_{11}&&&\mu_{10}&&&&&&&&&&&&&&1& \\
&&&\mu_{11}&&&&&&&&&&&&&&&1
\end{array}\right)
\end{array}
$$

Claim. The ring of invariants of this linear action, $\mathcal{O}(V)^H$, is not finitely generated as F-algebra.

Denoting the factors of H that are all isomorphic to $\mathbb{G}_a$ as follows: $H = G \times G_1 \times G_2 \times \cdots \times G_{11}$ we proceed to determine the invariants successively:

$$\mathcal{O}(V)^H = (\ldots((\mathcal{O}(V)^{G_{11}})^{G_{10}})\ldots)^G.$$

Lemma. *Let $\mu \in \mathbb{G}_a$ act on $\mathbb{A}^6$ via*

$$(\mu, (c, x, y, z, w_1, w_2)) \mapsto (c, x, y, z, w_1 + \mu c, w_2 + \mu q(x, y, z)),$$

where q is a non-constant polynomial in 3 variables.
Then $\mathcal{O}(\mathbb{A}^6)^{\mathbb{G}_a} = F[C, X, Y, Z, D]$ with $D = CW_2 - q(X, Y, Z)\, W_1$.

Proof. Suppose $f \in \mathcal{O}(\mathbb{A}^6)^{\mathbb{G}_a}$. Since f is constant on $\mathbb{G}_a$-orbits, for $c \neq 0$ we obtain $f(c,x,y,z,w_1,w_2) = f(c,x,y,z,0,d/c)$, which implies the identity

$$f(C,X,Y,Z,W_1,W_2) = f\left(C,X,Y,Z,0,\frac{D}{C}\right)$$
$$= f\left(C,X,Y,Z,0,W_2 - q(X,Y,Z)\frac{W_1}{C}\right).$$

This last expression can only be a polynomial with respect to C, if in each monomial occurring in f not involving the fifth variable the degree of the last variable is less than that of the first one. So in fact f is even a polynomial in C, X, Y, Z and D. □

We give the canonical coordinate functions of $V = F^{19}$ names reflecting the block structure of the H-action, namely:

$$C, X, Y, Z, S_1, S_2, S_3, T_1, T_2, T_3, U_1, U_2, U_3, V_1, V_2, V_3, V_4, V_5, V_6.$$

The last factor G_{11} is acting on the coordinate subspace W corresponding to C, X, Y, Z, V_5 and V_6 via

$$\begin{array}{ccc} G_{11} \times W & \longrightarrow & W \\ (\mu_{11}, (c,x,y,z,v_5,v_6)) & \longmapsto & (c,x,y,z,v_5+\mu_{11}\,c, v_6+\mu_{11}\,z) \end{array},$$

leaving the coordinate subspace corresponding to $S_1, S_2, \ldots, V_3, V_4$ pointwise fixed. So by the lemma, the invariants $\mathcal{O}(V)^{G_{11}}$ under G_{11} are generated over F by $C, X, Y, Z, S_1, S_2, S_3, T_1, T_2, T_3, U_1, U_2, U_3, V_1, \ldots, V_4$ and $V_5' := ZV_5 - CV_6$.

The group G_{10} acts on the corresponding space $\operatorname{Spec}(\mathcal{O}(V)^{G_{11}}) \simeq \mathbb{A}^{18}$ via

$$(\mu_{10}, (c,x,\ldots,v_3,v_4,v_5')) \mapsto (c,x,\ldots,v_3,v_4+\mu_{10}\,c, v_5'+\mu_{10}\,z^2),$$

and by the lemma, the invariants of this action are generated by $C, X, Y, \ldots, V_3$ and $V_4' := Z^2V_4 - CV_5'$.

On the corresponding space $\operatorname{Spec}(\mathcal{O}(V)^{G_{11}\times G_{10}}) \simeq \mathbb{A}^{17}$ the group G_9 acts via:

$$(\mu_9, (c,x,y,\ldots,v_2,v_3,v_4')) \mapsto (c,x,y,\ldots,v_2,v_3+\mu_9 c, v_4'+\mu_9 yz^2),$$

with invariants generated by $C, X, Y, \ldots, V_2$ and $V_3' := YZ^2V_3 - CV_4'$, as one can see by applying the lemma again.

Continuing in the same manner we finally obtain:

$$(\ldots((\mathcal{O}(V)^{G_{11}})^{G_{10}})\ldots)^{G_1} = F[C,X,Y,Z,S,T,U,V] = R[C],$$

where

$$S := X^2S_1 - CS_2' = X^2S_1 - C(XS_2 - CS_3) = X^2S_1 - CXS_2 + C^2S_3$$

$$T := Y^2T_1 - CYT_2 + C^2T_3$$

$$U := Z^2U_1 - CZU_2 + C^2U_3$$

$$V := XY^2Z^2V_1 - CY^2Z^2V_2 + C^2YZ^2V_3 - C^3Z^2V_4 + C^4ZV_5 - C^5V_6.$$

The group G acts on $\mathrm{Spec}(R[C]) \simeq \mathbb{A}^8$ as follows:

$$\begin{array}{ccc} \mathbb{G}_a \times \mathbb{A}^8 & \longrightarrow & \mathbb{A}^8 \\ (\lambda, p) = \left(\lambda, \begin{pmatrix} c \\ x \\ y \\ z \\ s \\ t \\ u \\ v \end{pmatrix} \right) & \mapsto & \begin{pmatrix} c \\ x \\ y \\ z \\ s + \lambda x^3 \\ t + \lambda y^3 \\ u + \lambda z^3 \\ v + \lambda (xyz)^2 \end{pmatrix} . \end{array}$$

Restricting the action to the subspace defined by $c = 0$, we find again the action introduced in the previous section, for $m = 2$. The ring of invariants $(\mathcal{O}(V))^H = (R[C])^G = R^G[C] = B[C]$ is the polynomial ring in one variable over the ring B considered by Roberts (for $m = 2$). (C is transcendental over R.) Therefore $(\mathcal{O}(V))^H$ is also not finitely generated as F-algebra.

Remark. For $m > 2$ there is an analogous construction of a linear action of $1 + 3 \cdot m + (3m - 1) = 6m$ copies of $\mathbb{G}_a$ acting on the vectorspace over F of dimension $4 + 3 \cdot (m + 1) + 3m = 6m + 7$, such that the ring of invariants is a polynomial ring in one variable over the ring B considered by Roberts, and hence not finitely generated as F-algebra.

REFERENCES

[LT] Lin Tan – An Elementary Counterexample to Hilbert's Fourteenth Problem. Manuscript, 1991.

[N] Nagata, M. – On the fourteenth problem of Hilbert. In: Proc. Int. Congress Math., 459–462 (1958), Cambridge University Press, 1960.

[R] Roberts, P. – An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem. J. Algebra **132**, 461–473 (1990).

[S] Steinberg, R. – Nagata's Example. Manuscript, 1991.